A SERIES FOR THE COLLISION PROBABILITY IN THE SHORT-ENCOUNTER MODEL

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ABSTRACT

A series to compute the collision probability of two spheres under the assumptions of short encounter has been derived. It is valid for both Gaussian and non-Gaussian distributions of the position. In the particular case of a Gaussian distribution the use of Hermite polynomials yields a simple form for the series. A region of practical interest has been carefully defined, and a sampling set of 244 cases was chosen. On this sampling set a comparison between the new series and previous algorithms has been performed for the Gaussian case. The presented series is faster than any other algorithm in every case. Numerical evidence suggests that if the series for the Gaussian case is truncated when the last term is smaller than the computed probability times a tolerance of 0.1, then the last term is an upper bound for the error. This article also presents very strong evidence for the case that the first two terms of the series are sufficient for the computation of the probability of collision and the absolute value of its second term is an upper bound for the error made when using it.

Index Terms— Space debris, collision probability, evasive maneuver, non Gaussian, fast computation

1. INTRODUCTION

The solution to the problem of space debris requires cleaning up the space around the Earth [2], especially the low Earth orbits, and enforcing laws that require space vehicles to take care of their own removal. The post-mission disposal is performed either by transfering to a graveyard orbit, or by descending to a low altitude orbit. This can be achieved either by saving enough fuel to go down to a low altitude or by passive means, such as electrodynamic tethers [3].

An active satellite or space station may be impacted by another body and suffer total or partial mission failure, or even fragment into thousands of pieces as it has happened in the past (see for instance the 2009 Iridium-Cosmos collision [4]). Derelict satellites and rocket bodies, and other debris cannot be removed from crowded orbits until active debris-removal technology is mature. In the mean time in order to avoid collisions, the best that one can do is a collision avoidance maneuver when another body threatens to hit the satellite. Performing a collision avoidance maneuver costs fuel, and one has to compute the collision probability first and then do the evasive maneuver if it surpasses certain threshold (i. e. between 10^{-5} and 10^{-3}) [5, 6, 7]. These kinds of maneuvers are performed several times during the lifetime of a satellite: as an example Envisat executed 4 maneuvers in 2011 [8].

The computation of the collision probability between two encountering bodies has been the subject of research since at least 1992 [9]. The usual assumptions in this computation are that the relative motion is a straight line within the region where the collision could take place, that their positions are Gaussianly distributed and that the objects are spheres. Akella and Alfriend [10] described a solution of this problem using an integral over space and time to take into account the relative motion of the bodies. Another similar, but less known method is the one of Khutorovsky *et alii* [11], which is the only precedent that we know of the approach presented in this article.

However, under the assumptions stated above the problem can be reduced to a time-independent, two-dimensional spatial integral. This is achived by essentially projecting the problem on a plane. In [9] Foster and Estes presented a numerical method to compute the collision probability p with a fixed spatial discretization. A finer discretization increases the precision of the calculation, but at a high computational cost. Patera [12, 13] solved the problem in polar coordinates doing a first analytic integration along the radial direction, followed by a numerical integration over the angle. The accuracy of the result can be improved using a tighter tolerance in the numerical integration. Chan [14, pp. 63 - 97] proposed an analytic power series that solves the isotropic problem (this is, the relative position uncertainty is the same in every direction). The anisotropic case is approximated using the previous result. While arbitrary accuracy can be achieved in the isotropic case by taking additional terms, the error made by approximating the general case by the isotropic one will not vanish. Another possibility is to discretize the integral and write it as a power series, which is the approach used by Alfano [15]. Note that Alfano had to adjust heuristically one of the terms, so this is not a purely analytical method. There is a very recent work of Serra *et alii* [16] which is fully analytic. Some of its features will be mentioned in sections 6 and 7.

In section 2 of this article we state the problem. In section 3 we find an expansion valid for a general probability density function (pdf) ρ of the relative position. When ρ is a Gaussian we show in section 4 that the expansion takes a simple form which involves Hermite polynomials. In sections 5 through 7, for the Gaussian case, we compare the performances of our algorithm and those of other authors and find that our algorithm is the fastest. In section 6 we show that, for all practical matters, our expansion can be reduced to only two terms, the second of which is also a bound for the error. Conclusions are stated in section 8.

We note that the probability of collision computed here or by other authors can be applied to any two objects that may collide whenever the three above mentioned assumptions hold, for example in artillery or other instances.

2. STATEMENT OF THE PROBLEM

Let there be two spheres of radii R_1 and R_2 whose centers are independently distributed according to the pdf's $f_1(\mathbf{r}, t)$ and $f_2(\mathbf{r}, t)$, respectively. The as-

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sumption that the objects are spheres is a conservative approximation if these spheres are taken as the spherical envelopes of the bodies that might collide. In a more exact approximation the objects could be substituted by their convex envelope and then consider Minkowski sums of their convex envelopes, which are easier to compute than the Minkowski sum of non convex bodies [17, Chapter 3: Minkowski addition].

There is an encounter plane (also called "b-plane") that contains the expected position of the two spheres at the expected time of closest approach, t_c. This plane is perpendicular to the expected relative velocity at that time.

In order to define this formally, we denote the expected value by $\langle\ \rangle$, that is

$$\langle \mathbf{r} \rangle_{1,2}(t) \equiv \int \mathrm{d}^3 r\, f_{1,2}(\mathbf{r}, t)\, \mathbf{r}. \tag{1}$$

Then t_c is the root of the equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\langle \mathbf{r}\rangle_2(t_c) - \langle \mathbf{r}\rangle_1(t_c)\right)^2 = \\ \left(\langle \mathbf{r}\rangle_2(t_c) - \langle \mathbf{r}\rangle_1(t_c)\right) \frac{d}{dt}\left(\langle \mathbf{r}\rangle_2(t_c) - \langle \mathbf{r}\rangle_1(t_c)\right) = 0. \end{aligned} \tag{2}$$

The encounter plane is perpendicular to the expected value of the relative velocity $\frac{\mathrm{d}}{\mathrm{d}t}\left(\langle \mathbf{r}\rangle_2(t) - \langle \mathbf{r}\rangle_1(t)\right)(t_c)$ and contains the centers of the two spheres at the time t_c, $\langle \mathbf{r}\rangle_1(t_c)$ and $\langle \mathbf{r}\rangle_2(t_c)$.

We denote by ρ_1 and ρ_2 the projections of $f_1(\mathbf{r}, t)$ and $f_2(\mathbf{r}, t)$ on the encounter plane, that is, the marginal pdf's resulting from integrating the original pdf's along the directions parallel to the expected value of the relative velocity. This reduces the three-dimensional collision/non-collision problem of two spheres to the two-dimensional overlap/non-overlap problem of two circles, which are the projection of the spheres. This simplification holds if the relative motion does not deviate significantly from a straight line for the duration of the encounter, and if the uncertainty of the relative velocity is negligible. This is called the short-encounter model [9, 10, 12, 14, 15, 16, 18, 19].

There is an approximation for the collision probability of these two spheres, which is [11]

$$p \approx \pi (R_1 + R_2)^2 \int \mathrm{d}^2 r\, \rho_1(\mathbf{r}) \rho_2(\mathbf{r}). \tag{3}$$

In the above expression and henceforth, the domain of integration is always $\mathbb{R}^2$ and the vectors are always in $\mathbb{R}^2$, unless otherwise stated. The above approximation is good when ρ_1 or ρ_2 do not vary significantly over the distance $R_1 + R_2$.

3. THE EXPANSION

The probability that two circles of radii R_1 and R_2 whose centers are independently distributed according to ρ_1 and ρ_2 overlap is

$$\begin{aligned} p = \int \mathrm{d}\mathbf{r_1}\, \rho_1(\mathbf{r_1}) \int_{|\mathbf{r_2} - \mathbf{r_1}| < R_1 + R_2} \mathrm{d}\mathbf{r_2}\, \rho_2(\mathbf{r_2}) = \\ \int \mathrm{d}\mathbf{r_2}\, \rho_2(\mathbf{r_2}) \int_{|\mathbf{r_1} - \mathbf{r_2}| < R_1 + R_2} \mathrm{d}\mathbf{r_1}\, \rho_1(\mathbf{r_1}). \end{aligned} \tag{4}$$

If ρ_2 is analytic, then it can be expanded in power series:

$$\begin{aligned} p = \int \mathrm{d}\mathbf{r_1}\, \rho_1(\mathbf{r_1}) \int_{|\mathbf{r_2} - \mathbf{r_1}| < R_1 + R_2} \mathrm{d}\mathbf{r_2} \\ \sum_{i=0}^{\infty} \frac{1}{i!} \sum_{j_1, \ldots, j_i = x, y} \frac{\partial^i \rho_2(\mathbf{r_1})}{\partial (\mathbf{r_1})_{j_1} \cdots \partial (\mathbf{r_1})_{j_i}} (\mathbf{r_2} - \mathbf{r_1})_{j_1} \cdots (\mathbf{r_2} - \mathbf{r_1})_{j_i}. \end{aligned} \tag{5}$$

When i is odd the integrals $\int_{|\mathbf{r_2} - \mathbf{r_1}| < R_1 + R_2} \mathrm{d}\mathbf{r_2}\, (\mathbf{r_2} - \mathbf{r_1})_{j_1} \cdots (\mathbf{r_2} - \mathbf{r_1})_{j_i}$ vanish due to the symmetry of the integration domain. We also suppose that the Taylor expansion of ρ_2 about any point $\mathbf{r_1}$ converges uniformly, so that the sum and integral signs may be commuted freely [20, Chapter 24: Uniform convergence and power series]. Then,

$$\begin{aligned} p = \sum_{i=0}^{\infty} \frac{1}{(2i)!} \int \mathrm{d}\mathbf{r_1}\, \rho_1(\mathbf{r_1}) \sum_{j_1, \ldots, j_{2i} = x, y} \frac{\partial^{2i} \rho_2(\mathbf{r_1})}{\partial (\mathbf{r_1})_{j_1} \cdots \partial (\mathbf{r_1})_{j_{2i}}} \\ \int_{|\mathbf{r_2} - \mathbf{r_1}| < R_1 + R_2} \mathrm{d}\mathbf{r_2}\, (\mathbf{r_2} - \mathbf{r_1})_{j_1} \cdots (\mathbf{r_2} - \mathbf{r_1})_{j_{2i}}. \end{aligned} \tag{6}$$

The integrals $\int_{|\mathbf{r_2} - \mathbf{r_1}| < R_1 + R_2} \mathrm{d}\mathbf{r_2}\, (\mathbf{r_2} - \mathbf{r_1})_{j_1} \cdots (\mathbf{r_2} - \mathbf{r_1})_{j_{2i}}$ are of the form $\int_{|\mathbf{r_2} - \mathbf{r_1}| < R_1 + R_2} \mathrm{d}\mathbf{r_2}\, (x_2 - x_1)^{2a} (y_2 - y_1)^{2b}$, where a and b are integers and the exponents $2a$ and $2b$ are even, otherwise the integral vanishes. They can be represented using [21, integral 3.621.5])

$$\int_0^{2\pi} \mathrm{d}\varphi\, \cos^{2a}\varphi\, \sin^{2b}\varphi = \frac{(2a-1)!!(2b-1)!!}{2^{a+b-1}(a+b)!}\, \pi. \tag{7}$$

Indeed,

$$\begin{aligned} \int_{|\mathbf{r_2} - \mathbf{r_1}| < R_1 + R_2} \mathrm{d}\mathbf{r_2}\, (x_2 - x_1)^{2a} (y_2 - y_1)^{2b} = \\ \int_0^{R_1 + R_2} d\ell\, \ell^{2i+1} \int_0^{2\pi} d\varphi\, \cos^{2a}\varphi\, \sin^{2b}\varphi = \\ \frac{(R_1 + R_2)^{2i+2}}{2i+2} \frac{(2a-1)!!(2b-1)!!}{2^{i-1} i!}\, \pi = \\ \frac{(R_1 + R_2)^{2i+2}}{2^i (i+1)!} (2a-1)!!(2b-1)!!\, \pi. \end{aligned} \tag{8}$$

When $j_1, \ldots, j_{2i}$ take the values x and y, the above integral appears $\binom{2i}{2b}$ times. Therefore,

$$\begin{aligned} p = \sum_{i=0}^{\infty} \frac{1}{(2i)!} \int \mathrm{d}\mathbf{r_1}\, \rho_1(\mathbf{r_1}) \sum_{j=0}^{i} \binom{2i}{2j} \frac{\partial^{2i} \rho_2(\mathbf{r_1})}{\partial x^{2(i-j)} \partial y^{2j}} \\ \frac{(R_1 + R_2)^{2i+2}}{2^i (i+1)!} (2(i-j)-1)!!(2j-1)!!\, \pi = \\ \sum_{i=0}^{\infty} \left(\frac{R_1 + R_2}{2}\right)^{2i+2} \frac{4\pi}{(i+1)!\, i!} \\ \int \mathrm{d}\mathbf{r_1}\, \rho_1(\mathbf{r_1}) \sum_{j=0}^{i} \binom{i}{j} \frac{\partial^{2i} \rho_2(\mathbf{r_1})}{\partial x^{2(i-j)} \partial y^{2j}}. \end{aligned} \tag{9}$$

When the two positions $\mathbf{r_1}$ and $\mathbf{r_2}$ are distributed according to the densities ρ_1 and ρ_2, respectively, then the relative position $\Delta\mathbf{r} \equiv \mathbf{r_2} - \mathbf{r_1}$ is distributed according to the density

$$\rho(\Delta\mathbf{r}) = \int \mathrm{d}^2 r_1\, \rho_1(\mathbf{r_1}) \rho_2(\mathbf{r_1} + \Delta\mathbf{r}) = \int \mathrm{d}^2 r\, \rho_1(\mathbf{r} - \Delta\mathbf{r}) \rho_2(\mathbf{r}). \tag{10}$$

It is convenient to use the convolution product notation,

$$f \otimes g(\mathbf{r}) \equiv \int \mathrm{d}\mathbf{r}''\, f(\mathbf{r}') g(\mathbf{r} - \mathbf{r}'), \tag{11}$$

for the above expression and write:

$$\rho(\Delta\mathbf{r}) = ((\rho_1 \circ (-1)) \otimes \rho_2)(\Delta\mathbf{r}), \tag{12}$$

where $(\rho_1 \circ (-1))(\mathbf{r_1}) \equiv \rho_1(-\mathbf{r_1})$. The function $\rho(\Delta\mathbf{r})$ is further discussed in [22, 23].

Collision takes place when the distance between the centers of the spheres is smaller than $R_1 + R_2$, thus

$$p = \int_{|\Delta\mathbf{r}| < R_1 + R_2} \mathrm{d}^2 \Delta r\, ((\rho_1 \circ (-1)) \otimes \rho_2)(\Delta\mathbf{r}). \tag{13}$$

It follows that Eq. (9) will yield the same result for any other couple (ρ_1', ρ_2') of pdf's such that $(\rho_1' \circ (-1)) \otimes \rho_2' = (\rho_1 \circ (-1)) \otimes \rho_2$. Furthermore, the probability of collision depends only on the relative distance, so that Eq. (9) will also yield the same result for any couple (ρ_1', ρ_2') of pdf's such that $(\rho_1'(\mathbf{r_1}), \rho_2'(\mathbf{r_2})) = (\rho_1(\mathbf{r_1} + \mathbf{r}'), \rho_2(\mathbf{r_2} + \mathbf{r}'))$, $\forall \mathbf{r}' \in \mathbb{R}^2$. We are going to use these two symmetries of $((\rho_1 \circ (-1)) \otimes \rho_2)(\Delta\mathbf{r})$ to simplify Eq. (9).

Similarly to the first section we use the notation $\langle \mathbf{r} \rangle_{1,2}(t) \equiv \int \mathrm{d}^3 r\, \rho_{1,2}(\mathbf{r}, t)\, \mathbf{r}$. The average of $(\rho_1 \circ (-1)) \otimes \rho_2$ is the difference of the averages of ρ_1 and ρ_2:

$$\begin{aligned} &\int \mathrm{d}^2\Delta r\, \Delta\mathbf{r} \int \mathrm{d}^2 r_1\, \rho_1(\mathbf{r_1})\, \rho_2(\mathbf{r_1} + \Delta\mathbf{r}) = \\ &\int \mathrm{d}^2\Delta r\, ((\Delta\mathbf{r} + \mathbf{r_1}) - \mathbf{r_1}) \int \mathrm{d}^2 r_1\, \rho_1(\mathbf{r_1})\, \rho_2(\mathbf{r_1} + \Delta\mathbf{r}) = \\ &\langle \mathbf{r} \rangle_2 - \langle \mathbf{r} \rangle_1. \end{aligned} \tag{14}$$

In the choice $(\rho_1'(\mathbf{r_1}), \rho_2'(\mathbf{r_2})) = (\delta(\mathbf{r_1} + (\langle \mathbf{r} \rangle_2 - \langle \mathbf{r} \rangle_1)), ((\rho_1 \circ (-1)) \otimes \rho_2)(\mathbf{r_2} + (\langle \mathbf{r} \rangle_2 - \langle \mathbf{r} \rangle_1)))$ one of the spheres is fixed at the location $\langle \mathbf{r} \rangle_1 - \langle \mathbf{r} \rangle_2$, while the other sphere is, on the average, at the origin but has gathered all the position uncertainty of the two original spheres. This choice yields the following simplification:

$$p = \sum_{i=0}^{\infty} \left(\frac{R_1 + R_2}{2} \right)^{2i+2} \frac{4\pi}{(i+1)(i!)^2} \sum_{j=0}^{i} \binom{i}{j} \frac{\partial^{2i} \rho(\langle \mathbf{r} \rangle_1 - \langle \mathbf{r} \rangle_2)}{\partial x^{2(i-j)} \partial y^{2j}}, \tag{15}$$

where $\rho \equiv (\rho_1 \circ (-1)) \otimes \rho_2$.

4. THE GAUSSIAN CASE

When both ρ_1 and ρ_2 are Gaussians, $(\rho_1 \circ (-1)) \otimes \rho_2$ is a Gaussian G whose covariance matrix is the sum of the covariance matrices of ρ_1 and ρ_2 and whose average is the difference of their averages. We choose the principal axes of G as axes of coordinates. In these axes let σ_x and σ_y be the standard deviations and let (x_0, y_0) be the coordinates of $\mathbf{r_0} \equiv \langle \mathbf{r} \rangle_1 - \langle \mathbf{r} \rangle_2$. Then

$$\frac{\partial^{2i} G(\mathbf{r_0})}{\partial x^{2(i-j)} \partial y^{2j}} = \frac{H_{2(i-j)}(x_0/\sigma_x)\, H_{2j}(y_0/\sigma_y)}{\sigma_x^{2(i-j)} \sigma_y^{2j}}\, G(\mathbf{r_0}), \tag{16}$$

where

$$G(\mathbf{r_0}) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{ -\frac{1}{2} \left(\frac{x_0^2}{\sigma_x^2} + \frac{y_0^2}{\sigma_y^2} \right) \right\} \tag{17}$$

and H_n is the n-th probabilists' Hermite polynomial, defined by $H_n(x) = (-1)^n \exp\left\{\frac{x^2}{2}\right\} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \exp\left\{-\frac{x^2}{2}\right\}$ [24, Chapter 8: Special functions] (not to be confused with the physicists' Hermite polynomial [21, Chapter 2: The classical orthogonal polynomials]; see also the Wikipedia article "Hermite polynomials" [1]).

Finally,

$$\begin{aligned} p = \sum_{i=0}^{\infty} p_i = \sum_{i=0}^{\infty} \left(\frac{R_1 + R_2}{2} \right)^{2i+2} \frac{4\pi}{(i+1)(i!)^2} \\ \sum_{j=0}^{i} \binom{i}{j} \frac{H_{2(i-j)}(x_0/\sigma_x)\, H_{2j}(y_0/\sigma_y)}{\sigma_x^{2(i-j)} \sigma_y^{2j}}\, G(\mathbf{r_0}), \end{aligned} \tag{18}$$

where p_i is the i-th term.

5. REGION OF INTEREST AND ITS SAMPLING

The more intuitive way to think is that pdf's of the errors in the position have to be convoluted and then projected on the encounter plane. But convolution and projections commute (see [25, section 9] or [26, section 10]), and from a geometrical point of view it is easier to think of the projection first and then consider the convolution of the 2-dimensional projections. To each Gaussian there is an ellipsoid naturally associated to its covariance matrix. The axes of this ellipsoid are parallel to the principal directions of the matrix and their lengths are proportional to the square roots of the eigenvalues of the covariance matrix. We are going to speak about this ellipsoid rather than about its covariance matrix. The projection of an ellipsoid of semiaxes $A < B < C$ is an ellipse whose semiaxes $a < b$ satisfy $A < a < b < C$. The convolution of two 2-dimensional Gaussians of semiaxes $a_1 < b_1$ and $a_2 < b_2$ is another Gaussian whose semiaxes $a < b$ satisfy $a_1 + a_2 < a < b < b_1 + b_2$. Therefore, given two error Gaussians of ellipsoids $A_1 < B_1 < C_1$ and $A_2 < B_2 < C_2$, the projection of the error Gaussian of the relative position onto any plane is an ellipse of semiaxes $a < b$ such that $A_1 + A_2 < a < b < C_1 + C_2$.

We take the combined radius as the unit of length (this is, $R_1 + R_2 = 1$). Based on the values reported in [16, 19, 27], we take the semiaxes of the error ellipsoids to range between 64 and 2048 along the velocity direction, and between 2 and 128 along the directions perpendicular to the velocity. Clearly the projections of the error ellipsoids of each object are ellipses whose semiaxes range between 2 and 2048. Their convolution is an ellipse of semiaxes between 4 and 4096. We now answer the following question: Can each of the semiaxes of the ellipse take any value between 4 and 4096 independently of the value of the other semiaxis? It could be that there is no way to obtain, say, an ellipse of semiaxes 4096 and 4096 by a projection and a convolution of the error ellipsoids. We shall see that this is actually the case and that only some combinations of the values of the semiaxes are possible.

Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be the expected velocities (in an inertial frame of origin at the Earth's center) of the objects which might collide. The long axes of the error ellipsoids are parallel to $\mathbf{v_1}$ and $\mathbf{v_2}$. These velocities determine the encounter plane, which is perpendicular to $\mathbf{v_1} - \mathbf{v_2}$. The intersection between the plane of Fig. 1 and the encounter plane has been labeled *lp*. We shall call *sp* the direction of the encounter plane perpendicular to *lp*. Then the projections of the longer error ellipsoid semiaxes are always along *lp*, as shown in the figure. Thus the projections along *lp* range from 2 to 2048, while the projections along *sp* range from 2 to 128. Therefore the error ellipse of the relative positions has one semiaxis which ranges from 4 to 4096 (*lp*) and another semiaxis which ranges from 4 to 256 (*sp*). In order to sample this region we go from 4 to 4096 and from 4 to 256 in powers of 4. This gives the set 4, 16, 64, 256, 1024, 4096 for the first case and the set 4, 16, 64, 256 for the second. This gives 18 different error ellipses.

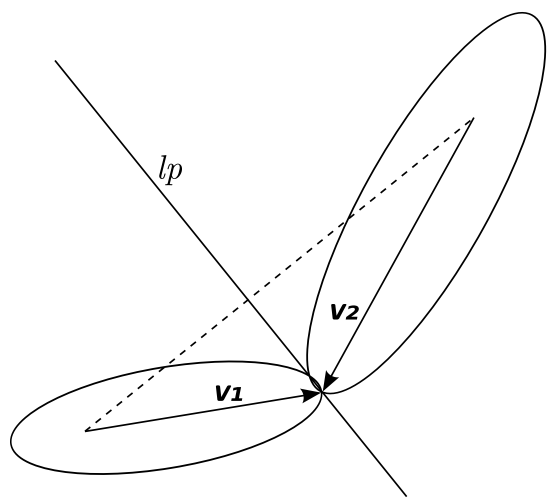

Fig. 1. The encounter plane is perpendicular to $\mathbf{v_1} - \mathbf{v_2}$ (dashed segment).

We have found the range of ellipses on the encounter plane to which the Eq. (18) giving the probability of collision is applied. But there are other parameters in the formula: the coordinates of the relative position. Eq. (18) is not applied to any possible combination of parameters but only to those for which an evasive maneuver is a non obvious possibility. For combinations of parameters for which a collision is very unlikely it is not necessary to compute the probability, since a first screening can be done using cruder methods [28, 29, 30]. Therefore computations are necessary only when the probability of collision may surpass some predetermined value at which an evasive maneuver is performed.

[1] `https://en.wikipedia.org/wiki/Hermite_polynomials`, accessed on September 10, 2015

These threshold values are usually between 10^{-3} and 10^{-5} (see [5, 6, 7]). These values depend on issues such as economic considerations, international space legislation and insurance companies' fees. However, the computation of the collision probability is also done when the decision to make an evasive maneuver has been taken [6]. When this happens the maneuver is required to yield a very small collision probability. A wider range of collision probabilities has also been considered before [31]. For these reasons we will study the integer powers of 10 in the range 10^{-1} to 10^{-7} [31].

For each of these values of constant p $(10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7})$ there is a curve of relative positions (x_0, y_0) for which the probability of collision takes these values. This curve is symmetric with respect to the x_0 and y_0 axes. We take four points on this curve as shown in Fig. 2: two along the axes of the ellipse $(x_0, y_0) = (x_{01}, 0)$ and $(x_0, y_0) = (0, y_{04})$ and two others for which $x_{02} = 2x_{01}/3$ and $x_{03} = x_{01}/3$. At first sight this would yield $18 \times 7 \times 4 = 504$ sample points. However, for some of the error ellipses and values of p, the curve described above is the empty set: no relative position can yield that collision probability. In the end there are only 244 cases left.

The interested reader is referred to the tables 1-5 of [1] for the numerical values for these cases.

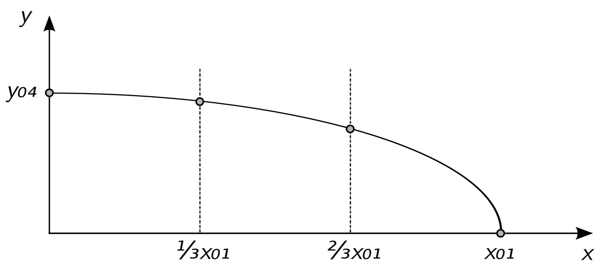

Fig. 2. Selected points in each curve of constant probability.

6. NUMBER OF TERMS NECESSARY FOR A GIVEN ACCURACY

Serra et alii [16] provide an algorithm which gives the number of terms necessary for a given accuracy in the computation of p. Other authors [14, 15] give simple recipes to estimate the necessary number of terms based on the error ellipse semiaxes, the relative position and the combined radius.

The expansion in Hermite polynomials converges everywhere, because, as remarked in section 3, the Taylor expansion of the Gaussian converges uniformly everywhere. There are, however, regions in which the terms change sign and become very large. This demands a very high precision and is a numerical drawback. (In Serra et alii [16] an ingenious method, *the preconditioning*, is presented to circumvent this problem). This happens, for example, when the ellipse of the Gaussian distribution is much smaller than combined body. But these regions of the parameter space are not regions of practical interest.

In the computation of the collision probability the uncertainty of the input is large. The pdf's of the position are assumed to be Gaussians for general reasons related to the central limit theorem and because of the analytic properties of the Gaussians, but it is not known to what extent they really are Gaussians. The spheres are, as stated in the introduction, nothing but quite conservative approximations to the actual shapes of the objects which may collide. Therefore we use a *tolerance* of 10%, i. e., we allow a relative error of 0.1 in p. For example if a probability of 0.0001 is obtained, we assure that $p \in [0.00009, 0.00011]$ for the given input. In other words, the absolute error is smaller than $0.0001/10$. Since the spherical approximation to the shape of the objects is so conservative, we know that the collision probability is actually $\lesssim 0.0001$.

For the threshold probabilities p used in our sampling we are demanding absolute errors of at most $p/10$. In the 244 examples which we have used to sample the region of interest, if we stop summing the series (Eq. (18)) when the last term is smaller than the admissible absolute error then the error is much smaller (between 27 and $56,000$ times smaller) than the last term neglected. This is very strong evidence for the case that the last term can be used as a bound for the absolute error in the region of interest.

In all the considered cases, the first two terms of the series (Eq .(18)) were necessary and sufficient to guarantee a 10% tolerance, that is

$$p \approx p_0 + p_1 = \frac{2}{\sigma_x \sigma_y} \exp\left\{-\frac{1}{2}\left(\frac{x_0^2}{\sigma_x^2} + \frac{y_0^2}{\sigma_y^2}\right)\right\} \left(\frac{R_1 + R_2}{2}\right)^2 \left(1 + \frac{1}{2}\left(\frac{\left(\frac{x_0}{\sigma_x}\right)^2 - 1}{\sigma_x^2} + \frac{\left(\frac{y_0}{\sigma_y}\right)^2 - 1}{\sigma_y^2}\right)\left(\frac{R_1 + R_2}{2}\right)^2\right), \tag{19}$$

and the absolute error is bounded by

$$|p_1| = \frac{1}{\sigma_x \sigma_y} \exp\left\{-\frac{1}{2}\left(\frac{x_0^2}{\sigma_x^2} + \frac{y_0^2}{\sigma_y^2}\right)\right\} \left(\frac{R_1 + R_2}{2}\right)^4 \left|\frac{\left(\frac{x_0}{\sigma_x}\right)^2 - 1}{\sigma_x^2} + \frac{\left(\frac{y_0}{\sigma_y}\right)^2 - 1}{\sigma_y^2}\right|. \tag{20}$$

7. COMPARISON WITH OTHER AUTHORS

In order to compute p and the number of terms needed for a given accuracy, different authors developed analytical and numerical methods [9, 13, 14, 15, 16]. Of these methods, only the ones by Chan (for the isotropic cases only), by Serra et alii and by ourselves are completely analytic [2]. Of the last two, our method is simpler to derive and can be applied for non Gaussian pdf's of the position. Only Serra et alii present a completely analytic way to determine the number of terms used to find p with a given accuracy.

All these algorithms can compute p with arbitrarily high accuracy. If computing speed were not an issue, one would choose the simplest or most pleasing method from a mathematical point of view, and some comments in that regard are written in the preceding paragraph. But from a practical point of view, speed is the first concern. Consequently, comparing the speed of the different methods is an important criterion according to which we shall compare them. Other authors have compared different methods using different criteria [16, 31].

We implemented all the algorithms (Foster, Patera, Alfano, Chan, Serra, and the proposed method) in Fortran 95, since in order to compare speed, it is convenient to use a low-level computing language like Fortran or C. Then, we compared the performance of the methods on an Intel(R) Core(TM) i5-2467M CPU at 1.60GHz machine running Ubuntu 14.04 and the GNU Fortran 4.8.3-2 compiler.

Figure 3 shows for each method the time needed to compute each of the 244 cases described above with a relative error smaller than 10%, as advertised in the previous section. Eq. (19) is the fastest in each of the 244 cases. Similar but slightly longer times are obtained using Serra's method and Chan's method, but the latter does not achieve the required accuracy in some of the cases (number 175, 176, 179, 180, 183 and 194 of tables 4 and 9 of [1]). An erratic behaviour can be seen in the computation time of Alfano's Method in Fig 3, which is caused by the disparity in performance of the Fortran intrinsic function *erf* (Error function) for different values of the input.

8. CONCLUSIONS

A series to compute the collision probability of two spheres under the assumptions of short encounter has been derived. It is valid for both Gaussian and non-Gaussian distributions of the position. In the particular case of a

[2] Note that when applying the algorithm 1 of Serra et alii [16], one must correct the numerators in the c_2 and c_3 coefficients: they should be 12 and 144 (instead of 6 and 24, respectively).

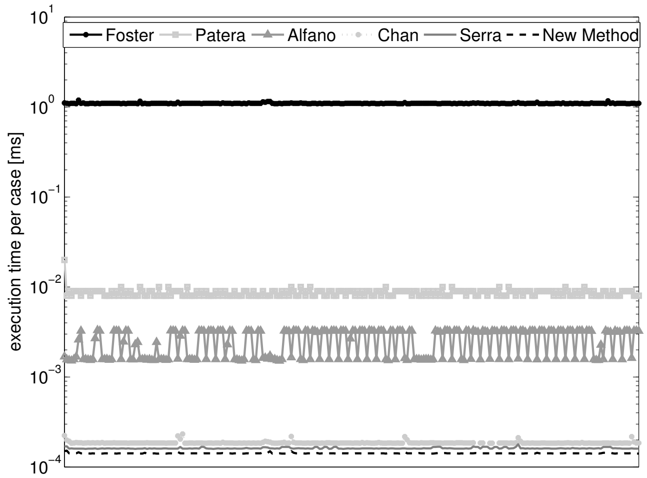

Fig. 3. Comparison of the execution time for the 244 different cases using different methods.

Gaussian distribution the use of Hermite polynomials yields a simple form for the series.

The region in which the parameters of the problem (probability density function of the relative position, relative position of the spheres, size of spheres) define cases of practical interes t has been carefully defined. A sampling set of 244 cases in this region has been chosen. On this sampling set a comparison between our series and previous algorithms has been performed for the Gaussian case. Our series is faster than any other algorithm in every case. Numerical evidence strongly suggests that if the series for the Gaussian case is truncated when the last term is smaller than the computed probability times a tolerance of 0.1, then the last term is an upper bound for the error.

The series presented here can yield arbitrary precision. But from a practical point of view, this article presents very strong evidence for the case that, given the current lack of precision in the knowledge of the error of the position of the colliding objects, its first two terms alone are sufficient for the computation of the probability of collision; and the absolute value of its second term is an upper bound for the error made when using it.

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